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ON THE COHOMOLOGY OF SPLIT EXTENSIONS OF FINITE GROUPS

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Abstract. Let $G = H \rtimes Q$ be a split extension of finite groups. A theorem of Charlap and Vasquez gives an explicit description of the differentials d_2 in the Lyndon-Hochschild-Serre spectral sequence of the extension with coefficients in a field k. We generalize this to give an explicit description of all the d_r $(r \geq 2)$ in this case. The generalization is obtained by associating to the group extension a new twisting cochain, which takes values in the kG-endomorphism algebra of the minimal kH-projective resolution induced from H to G. This twisting cochain not only determines the differentials, but also allows one to construct an explicit kG-projective resolution of k.

1. Introduction

Let $G = H \rtimes Q$ be a split extension of finite groups, k a field, and consider the Lyndon-Hochschild-Serre (LHS) spectral sequence $E_2^{**} \cong H^*(Q, H^*(H,k)) \Rightarrow H^*(G,k)$. In a previous paper, we gave a simplified proof of a modified version of a theorem of Charlap and Vasquez which allows one to explicitly compute the differentials d_2 in this case, and applied this result to a problem in the cohomology of extraspecial groups (see [4, 5] and [10, Theorem 1]). Here, we generalize the Charlap-Vasquez theorem to provide a method for explicitly computing all the differentials d_r $(r \geq 2)$.

The Charlap-Vasquez method for d_2 proceeds as follows. Let $P \to k$ be the minimal kH-projective resolution, and for each $\sigma \in Q$, let $\nu_1[\sigma]\colon P \to P$ be a chain map which commutes with the augmentation and is σ-linear, i.e. a linear map satisfying $\nu_1[\sigma](hx) = \sigma h \sigma^{-1} \nu_1[\sigma](x)$ $(h \in H,\ x \in P)$. The existence of these maps follows from the Comparison Theorem of Homological Algebra (cf. [2, Theorem 2.4.2 and Remark]). Moreover, the uniqueness part of the Comparison Theorem says that for each $\sigma, \tau \in Q$, there is a $\sigma\tau$-linear map $\nu_2[\sigma|\tau]\colon P \to P$ which raises degree by one and which satisfies

$$\partial \circ \nu_2[\sigma|\tau] + \nu_2[\sigma|\tau] \circ \partial = \nu_1[\sigma\tau] - \nu_1[\sigma] \circ \nu_1[\tau]. \tag{1.1}$$

The Charlap-Vasquez Theorem is an expression for d_2 in terms of the maps $\nu_2[\sigma|\tau]$.

As a corollary to our main Theorem 8.1 we obtain a direct generalization of the above. It reduces the calculation of the differentials $d_2, \ldots, d_r$ to the computation of certain maps $\nu_n[\sigma_1|\cdots|\sigma_n]\colon P \to P$, one for each $1 \leq n \leq r$ and $\sigma_1, \ldots, \sigma_n \in Q$.

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Each $\nu_n[\sigma_1|\cdots|\sigma_n]$ is required to be $\sigma_1\cdots\sigma_n$-linear and to raise degree by $n-1$, and together they must satisfy

$$\begin{aligned}(1.2)\quad \partial\circ\nu_n[\sigma_1|\cdots|\sigma_n] &+ (-1)^n\nu_n[\sigma_1|\cdots|\sigma_n]\circ\partial \\ &= \sum_{i=1}^{n-1}(-1)^{i+1}\nu_{n-1}[\sigma_1|\cdots|\sigma_i\sigma_{i+1}|\cdots|\sigma_n] \\ &\quad + \sum_{i=1}^{n-1}(-1)^i\nu_i[\sigma_1|\cdots|\sigma_i]\circ\nu_{n-i}[\sigma_{i+1}|\cdots|\sigma_n].\end{aligned}$$

Notice that when $n=2$, equation (1.2) reduces to equation (1.1). The map ν_2 can be interpreted as an obstruction to finding an action of Q on P which is consistent with the action on H, and the maps ν_n for $n\geq 2$ can be viewed as higher-level obstructions. In particular, if Q really does act on P, then all these obstructions vanish, and $E_2=E_\infty$. (In fact a stronger result is true, namely $E_2\cong H^*(G,k)$ as graded rings, cf. Evens [6, Section 2.5].) Another well-known result which follows from the main theorem is that the differentials into the horizontal edge vanish (cf. [6, Proposition 7.3.2]). Both of these results are discussed in further detail in Section 8.

The generalization of the Charlap-Vasquez theorem, as expressed above, requires the kQ-bar resolution of k to be used in constructing the LHS spectral sequence. But our main theorem actually works with a wider class of free resolutions. We call these *special resolutions,* and define them axiomatically in Section 4. This class includes the bar resolution, the reduced bar resolution, and the minimal resolution of an elementary abelian 2-group in characteristic 2. This last case makes our techniques particularly efficient for group extensions of the form $H\rtimes(\mathbb{Z}/2)^n$, which we will demonstrate in a subsequent paper.

The key to the generalization is the use of a *twisting cochain* (cf. Brown [3]), which is a map t from a differential graded coalgebra $\mathfrak{C}$ to a differential graded algebra $\mathfrak{A}$ satisfying (among other conditions) $d(t)+t\cup t=0$. This condition is designed specifically to allow one to use t to perturb the standard differential in the tensor product of a differential graded $\mathfrak{C}$-comodule with a differential graded $\mathfrak{A}$-module. In our case, $\mathfrak{C}$ is the special resolution for Q tensored over kQ with k, and $\mathfrak{A}$ is the kG-endomorphism algebra of P induced from H to G. Our twisting cochain, which is shown to be unique in an appropriate sense, captures many aspects of the cohomology of the extension. Not only does it produce the differentials in the spectral sequence, but it can be used to construct an explicit projective resolution of k over kG; in essence, this is an explicit version of a theorem of C. T. C. Wall [12].

The material is organized as follows. In Section 2 we establish our notation and review the homological algebra that we require; the propositions there are easy exercises in the definitions. In Section 3 we establish some basic facts about twisting cochains that will be fundamental for the main theorem; in particular we prove an analogue of the Comparison Theorem. Section 4 deals with the class of special resolutions mentioned above. In Section 5 we define the twisting cochain associated to an arbitrary (not necessarily split) extension, and in Section 6, which is not required for the main theorem, we show how this twisting cochain can be used to construct a kG-projective resolution of k. In Section 7 we add the assumption that the extension is split, and derive an explicit form for the twisting cochain in this case. The main theorem on the differentials is stated and proved in Section 8. We conclude in Section 9 with some small examples to demonstrate how the theorem can be applied; in a future paper we will apply the theory to more complex spectral sequences of extensions of elementary abelian 2-groups, where little had previously been known.

2. Background from homological algebra

Complexes. Let R be a commutative ring and Λ an R-algebra. (By *algebra* we always mean an associative algebra with unit, and similarly coalgebras are assumed to be coassociative with counit.) If C is a complex of left or right Λ-modules, let $Z_nC = \operatorname{Ker}(\partial\colon C_n \to C_{n-1})$, $B_nC = \operatorname{Im}(\partial\colon C_{n-1} \to C_n)$, and $H_nC = Z_nC/B_nC$. If C, D are complexes of right, left Λ-modules respectively, let $C \otimes_\Lambda D$ denote the complex with $(C\otimes_\Lambda D)_n = \bigoplus_{p+q=n} C_p \otimes_\Lambda C_q$, with differential given by $\partial(x\otimes y) = \partial(x)\otimes y + (-1)^p x\otimes \partial(y)$ for $x \in C_p, y \in D_q$.

If C and D are both complexes of left Λ-modules, then let $\operatorname{Hom}_\Lambda(C, D)$ denote the complex with $\operatorname{Hom}_\Lambda(C, D)_n = \prod_{q=p+n} \operatorname{Hom}_\Lambda(C_p, D_q)$, with differential given by $\partial(f) = \partial\circ f - (-1)^n f\circ\partial$, for $f \in \operatorname{Hom}_\Lambda(C, D)_n$. If $f - g \in B_n \operatorname{Hom}_\Lambda(C, D)$ we write $f \simeq g$ (and say f and g are *chain homotopic*). We also define a subcomplex $\hom_\Lambda(C, D)$ of $\operatorname{Hom}_\Lambda(C, D)$, which is equal to the latter in positive degrees, $Z_0 \operatorname{Hom}_\Lambda(C, D)$ in degree 0, and 0 in negative degrees. In particular, $f \in \hom_\Lambda(C, D)_0$ if, and only if, f is a Λ-chain map, i.e. f is a Λ-homomorphism which preserves degree and $\partial\circ f = f\circ\partial$. If B is a third complex of left Λ-modules and $f \in \operatorname{Hom}_\Lambda(B, C)_p$, then we define $f^*\colon \operatorname{Hom}_\Lambda(C, D) \to \operatorname{Hom}_\Lambda(B, D)$ by $f^*(g) = (-1)^{pq} g\circ f$ for g homogeneous of degree q. If $g \in \operatorname{Hom}_\Lambda(C, D)$, then we define $g_*\colon \operatorname{Hom}_\Lambda(B, C) \to \operatorname{Hom}_\Lambda(B, D)$ by $g^*(f) = g\circ f$. Of course, similar definitions apply to right modules. Finally, any chain complex may be considered a cochain complex, and *vice-versa*, by setting $C^n = C_{-n}$.

Proposition 2.1. *Let B, C, and D be complexes of left Λ-modules, and let $f_0 \in \hom_\Lambda(B, C)_0$ and $g_0 \in \hom_\Lambda(C, D)_0$. Then each of the following maps is a map of complexes:*

(i) $C\otimes_R D \to D\otimes_R C$, *defined by* $x\otimes y \mapsto (-1)^{\deg(x)\deg(y)} y\otimes x$,
(ii) $\operatorname{Hom}_\Lambda(C, D)\otimes_R C \to D$, *defined by* $f\otimes x \mapsto f(x)$,
(iii) $C\otimes_R \operatorname{Hom}_\Lambda(C, D) \to D$, *defined by* $x\otimes f \mapsto (-1)^{\deg(f)\deg(x)} f(x)$,
(iv) $\operatorname{Hom}_\Lambda(C, D)\otimes_R \operatorname{Hom}_\Lambda(B, C) \to \operatorname{Hom}_\Lambda(B, D)$, *defined by* $f\otimes g \mapsto f\circ g$,
(v) $\operatorname{Hom}_\Lambda(B, C)\otimes_R \operatorname{Hom}_\Lambda(C, D) \to \operatorname{Hom}_\Lambda(B, D)$, *defined by* $f\otimes g \mapsto (-1)^{\deg(f)\deg(g)} g\circ f$,
(vi) $(f_0)^*\colon \operatorname{Hom}_\Lambda(C, D) \to \operatorname{Hom}_\Lambda(B, D)$, *defined by* $g \mapsto g\circ f_0$,
(vii) $(g_0)_*\colon \operatorname{Hom}_\Lambda(B, C) \to \operatorname{Hom}_\Lambda(B, D)$, *defined by* $f \mapsto g_0\circ f$.

Now suppose X and Y are complexes of right Λ-modules. We define a map

$$\operatorname{Hom}_\Lambda(X, Y)\otimes_R \operatorname{Hom}_\Lambda(C, D) \to \operatorname{Hom}_R(X\otimes_\Lambda C, Y\otimes_\Lambda D) \qquad (2.1)$$
$$f\otimes g \mapsto f\times g$$

where $(f\times g)(x\otimes y) = (-1)^{\deg(x)\deg(g)} f(x)\otimes g(y)$ for homogeneous x, y, f, g. We then have

Proposition 2.2. *Let B, C, and D be complexes of left Λ-modules, X, Y, and Z complexes of right Λ-modules, $f \in \operatorname{Hom}_\Lambda(B, C)_r$, $g \in \operatorname{Hom}_\Lambda(X, Y)_s$, $f' \in \operatorname{Hom}_\Lambda(C, D)_{r'}$, and $g' \in \operatorname{Hom}_\Lambda(Y, Z)_{s'}$. Then*

(i) $\partial \circ (f \times g) = (\partial \circ f) \times g + (-1)^r f \times (\partial \circ g)$,
(ii) $(f \times g) \circ \partial = f \times (g \circ \partial) + (-1)^s (f \circ \partial) \times g$,
(iii) $\partial(f \times g) = \partial(f) \times g + (-1)^r f \times \partial(g)$,
(iv) $(f' \times g') \circ (f \times g) = (-1)^{rs'} (f' \circ f) \times (g' \circ g)$.

Cup products. We now restrict our attention to the case where Λ is a Hopf algebra with antipode over a field k. All of our Λ-modules will be left Λ-modules, but these can always be considered right Λ-modules (and *vice-versa*), because of the antipode. All unlabeled tensor products are assumed to be tensor products over k.

Generally, cup products on the level of complexes arise as follows. Let $A, B, \dots, F$ be chain complexes of Λ-modules, and let $\Delta\colon E \to A \otimes C$ and $\mu\colon B \otimes D \to F$ be Λ-chain maps. The product map (2.1)

$$\mathrm{Hom}_k(A, B) \otimes \mathrm{Hom}_k(C, D) \xrightarrow{\times} \mathrm{Hom}_k(A \otimes C, B \otimes D)$$

restricts to a map

$$\mathrm{Hom}_\Lambda(A, B) \otimes \mathrm{Hom}_\Lambda(C, D) \xrightarrow{\times} \mathrm{Hom}_\Lambda(A \otimes C, B \otimes D), \tag{2.2}$$

which is also a map of complexes. We then define the cup product with respect to the "comultiplication" Δ and the "multiplication" μ to be the map

$$\begin{aligned} \mathrm{Hom}_\Lambda(A, B) \otimes \mathrm{Hom}_\Lambda(C, D) &\xrightarrow{\cup} \mathrm{Hom}_\Lambda(E, F) \\ f \otimes g &\longmapsto f \cup g = \mu \circ (f \times g) \circ \Delta. \end{aligned}$$

This is also a chain map, since it is just the composite of (2.2) followed by Δ^* then μ_*, each of which is a chain map, by Propositions 2.1 and 2.2.

Suppose A', B', C', D', E', F' form another set of complexes of Λ-modules with a multiplication and comultiplication map. Then there is a natural way to define a multiplication and comultiplication map on the pairwise tensor products. Namely, define Δ and μ as the composites

$$\Delta\colon\ E' \otimes E \xrightarrow{\Delta \times \Delta} A' \otimes C' \otimes A \otimes C \longrightarrow A' \otimes A \otimes C' \otimes C\ ,$$

$$\mu\colon\ B' \otimes B \otimes D' \otimes D \longrightarrow B' \otimes D' \otimes B \otimes D \xrightarrow{\mu \times \mu} F' \otimes F,$$

where the maps which transpose the two inner factors are defined by

$$w \otimes x \otimes y \otimes z \mapsto (-1)^{\deg(x)\deg(y)} w \otimes y \otimes x \otimes z.$$

By Propositions 2.1 and 2.2, these are chain maps. Hence there is a cup product

$$\mathrm{Hom}_\Lambda(A' \otimes A, B' \otimes B) \otimes \mathrm{Hom}_\Lambda(C' \otimes C, D' \otimes D) \xrightarrow{\cup} \mathrm{Hom}_\Lambda(E' \otimes E, F' \otimes F).$$

The cup product interacts nicely with the other operations, as in the following:

Proposition 2.3. *The following hold, wherever they are defined:*

(i) $\partial \circ (f \cup g) = (\partial \circ f) \cup g + (-1)^{\deg(f)} f \cup (\partial \circ g)$,
(ii) $(f \cup g) \circ \partial = f \cup (g \circ \partial) + (-1)^{\deg(g)} (f \circ \partial) \cup g$,
(iii) $\partial(f \cup g) = \partial(f) \cup g + (-1)^{\deg(f)} f \cup \partial(g)$,
(iv) $(f' \cup g') \circ (f \cup g) = (-1)^{\deg(f)\deg(g')} (f' \circ f) \cup (g' \circ g)$,
(v) $(f' \cup g') \times (f \cup g) = (-1)^{\deg(f)\deg(g')} (f' \times f) \cup (g' \times g)$.

Differential algebra. A *differential graded algebra* (DGA) over Λ is a complex $\mathfrak{A}$ of Λ-modules which is also a Λ-algebra, such that the unit map $k \xrightarrow{\eta} \mathfrak{A}$ (sending the unit of k to the unit of $\mathfrak{A}$) and the multiplication map $\mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\mu} \mathfrak{A}$ are maps of complexes. (Here k is considered as a complex concentrated in degree 0.) A morphism of DGAs is a map of complexes which is also a map of Λ-algebras. An *augmented DGA* is a DGA which is 0 in negative degrees together with a map $\epsilon\colon \mathfrak{A} \to k$ which is a map of DGAs. If $\mathfrak{A}$ is a DGA, then a *differential graded left* $\mathfrak{A}$*-module* (left DGM) is a complex M of left $\mathfrak{A}$-modules such that the multiplication map $\mathfrak{A} \otimes M \xrightarrow{\mu} M$ is a map of complexes. Right DGMs are defined similarly.

A *differential graded coalgebra* (DGCA) $\mathfrak{C}$ is defined dually. Hence $\mathfrak{C}$ is a complex of Λ-modules which is also a Λ-coalgebra such that the counit map $\mathfrak{C} \xrightarrow{\epsilon} k$ and the comultiplication map $\mathfrak{C} \xrightarrow{\Delta} \mathfrak{C} \otimes \mathfrak{C}$ are maps of complexes. A morphism of DGCAs, a *coaugmented DGCA*, and a *differential graded comodule* (DGCM) are also defined dually. We also use the "partition notation" for Δ: if $n = a_1 + \cdots + a_r \geq 1$, then $\Delta_{a_1^{i_1}, \ldots, a_r^{i_r}}$ denotes the map

$$\mathfrak{C}_n \xrightarrow{\qquad} \mathfrak{C} \xrightarrow{(1\times\cdots\times 1\times\Delta)\cdots(1\times\Delta)\Delta} \mathfrak{C}^{\otimes n} \xrightarrow{\qquad} \mathfrak{C}_{a_1}^{\otimes i_1} \otimes \cdots \otimes \mathfrak{C}_{a_r}^{\otimes i_r}.$$

It is immediate from the discussion above that for a DGA $\mathfrak{A}$ and a DGCA $\mathfrak{C}$ over Λ, there is a cup product

$$\operatorname{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A}) \otimes \operatorname{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A}) \xrightarrow{\cup} \operatorname{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A}),$$

and that this gives $\operatorname{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A})$ the structure of a DGA over k with unit $\eta\epsilon$. Moreover, if M is a left DGM for $\mathfrak{A}$ and N is a left DGCM for $\mathfrak{C}$, then there is a cup product

$$\operatorname{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A}) \otimes \operatorname{Hom}_\Lambda(N, M) \xrightarrow{\cup} \operatorname{Hom}_\Lambda(N, M),$$

which gives $\operatorname{Hom}_\Lambda(N, M)$ the structure of a left DGM over $\operatorname{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A})$. A similar statement holds for right modules.

Spectral sequences. Let Λ be an arbitrary ring and $A = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}$ a bigraded left Λ-module. Let $\pi^{p,q}\colon A \to A^{p,q}$ denote the projection map, set $A^n = \bigoplus_{p+q=n} A^{p,q}$ and $F^pA^n = \bigoplus_{p'\geq p} A^{p',n-p'}$. Let $\delta\colon A \to A$ be a Λ-map satisfying $\delta(F^pA^n) \subseteq F^pA^{n+1}$ and $\delta^2 = 0$. Hence we can write $\delta = \delta_0 + \delta_1 + \ldots$, where $\delta_i(A^{p,q}) \subseteq A^{p+i,q-i+1}$, and the condition $\delta^2 = 0$ can be rewritten $\sum_{i=0}^n \delta_i\delta_{n-i} = 0$ ($n \geq 0$). We call (A, δ) a *bigraded complex*. For each $r \geq 0$, set

$$\begin{aligned}
Z_r^{p,q} &= \{x \in A^{p,q} \mid \exists y \in F^{p+1}A^{p+q} \text{ s.t. } \delta(x+y) \in F^{p+r}A^{p+q+1}\},\\
B_r^{p,q} &= \{\pi^{p,q}\delta(y) \mid y \in F^{p-r+1}A^{p+q-1},\ \delta(y) \in F^pA^{p+q}\},\\
E_r^{p,q} &= Z_r^{p,q}/B_r^{p,q}.
\end{aligned}$$

Of course $A = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq B_1 \supseteq B_0 = 0$, so the definition of E_r makes sense. Define the differentials $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ as follows: let $x \in Z_r^{p,q}$, and choose $y \in F^{p+1}A^{p+q}$ such that $\delta(x+y) \in F^{p+r}A^{p+q+1}$. The element

$$\pi^{p+r,q-r+1}\,\delta(x+y) + B_r^{p+r,q-r+1} \in E_r^{p+r,q-r+1}$$

is seen to be independent of the choice of y, so sending x to that element yields a well-defined map $Z_r^{p,q} \to E_r^{p+r,q-r+1}$. One checks that $B_r^{p,q}$ is in the kernel of this map, and defines the induced map to be d_r. Finally, it can be verified that $d_r^2 = 0$, $\operatorname{Ker}(d_r) = Z_{r+1}/B_r$, and $\operatorname{Im}(d_r) = B_{r+1}/B_r$. Hence, $H(E_r, d_r) \cong E_{r+1}$. We call the sequence (E_r, d_r) $(r \geq 0)$ the *spectral sequence of the bigraded complex* (A, δ). Note that if $\delta_i = 0$ for $i \geq 2$, then (A, δ_0, δ_1) is a double complex, and this spectral sequence is the familiar spectral sequence of a double complex.

3. Twisting cochains

Fundamentals. Let Λ be a Hopf algebra (with antipode) over a field k. As before, $\otimes$ means $\otimes_k$. Let $\mathfrak{A}$ be an augmented DGA over Λ with differential d, multiplication μ, unit map η, and augmentation ϵ. Let $\mathfrak{C}$ be a DGCA over Λ with differential d, comultiplication Δ, and counit ϵ.

Definition 3.1. A map $t \in \mathrm{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A})_{-1}$ is called a *twisting cochain* if $\epsilon t_1 = 0$ and $d(t) + t \cup t = 0$. Two twisting cochains $\tilde{t}$ and t are said to be *homotopic* if there is a map $s \in \mathrm{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A})_0$ such that $\epsilon s = \epsilon$ and $d(s) = s \cup \tilde{t} - t \cup s$, and if this is the case, one writes $s \colon \tilde{t} \simeq t$.

The conditions $d(t) + t \cup t = 0$ and $d(s) = s \cup \tilde{t} - t \cup s$ can be rewritten

$$d_{i-1}t_i + t_{i-1}d_i + \sum_{j=1}^{i-1} t_j \cup t_{i-j} = 0, \qquad i \geq 2, \tag{3.1}$$

$$d_i s_i - s_{i-1}d_i = \sum_{j=0}^{i-1} s_j \cup \tilde{t}_{i-j} - \sum_{j=0}^{i-1} t_{i-j} \cup s_j, \qquad i \geq 1, \tag{3.2}$$

respectively, since $t_j = 0$ for all $j \leq 0$. It can also be shown that under appropriate conditions, homotopy of twisting cochains is an equivalence relation (cf. [8, Lemma 1.4.1]), though we will not need this fact here. More important for our purposes will be the following analogy to the Comparison Theorem:

Proposition 3.2. *Let $\mathfrak{A}$ be an augmented DGA over Λ and $\mathfrak{C}$ a DGCA over Λ. Let $n \geq 1$ and assume $\mathfrak{C}_{n+1}$ is a projective Λ-module. Then*

- (i) *Suppose the complex $\cdots \to \mathfrak{A}_1 \to \mathfrak{A}_0 \xrightarrow{\epsilon} k$ is exact at degree $n-1$. Given Λ-homomorphisms $t_1, \ldots, t_n$ such that $\epsilon t_1 = 0$ and* (3.1) *holds for $2 \leq i \leq n$, there exists a Λ-homomorphism t_{n+1} such that* (3.1) *holds for $2 \leq i \leq n+1$.*
- (ii) *Suppose $\tilde{t}$ and t are twisting cochains and $\mathfrak{A}$ is exact at degree n. Given Λ-homomorphisms $s_0, \ldots, s_n$ such that $\epsilon s_0 = \epsilon$ and* (3.2) *holds for $1 \leq i \leq n$, there exists a Λ-homomorphism s_{n+1} such that* (3.2) *holds for $1 \leq i \leq n+1$.*

Proof. To prove (i) for $n \geq 2$ we need only show

$$d_{n-1} \circ \left(t_n d_{n+1} + \sum_{i=1}^{n} t_i \cup t_{n+1-i} \right) = 0. \tag{3.3}$$

The result then follows by exactness and the projectivity of $\mathfrak{C}_{n+1}$. However,

$$\begin{aligned}
d \circ \sum_{i=1}^{n} t_i \cup t_{n+1-i} &= \sum_{i=1}^{n} [dt_i \cup t_{n+1-i} - t_i \cup dt_{n+1-i}] \\
&= \sum_{i=1}^{n} [dt_i \cup t_{n+1-i} - t_i \cup dt_{n+1-i}] \\
&= \sum_{i=1}^{n} \left[\left(-t_{i-1}d - \sum_{j=1}^{i-1} t_j \cup t_{i-j} \right) \cup t_{n+1-i} + t_i \cup \left(t_{n-i}d + \sum_{j=1}^{n-i} t_j \cup t_{n-i-j+1} \right) \right] \\
&= \sum_{i=1}^{n-1} [t_i \cup t_{n-i}d - t_i d \cup t_{n-i}] \\
&= \left(\sum_{i=1}^{n-1} t_i \cup t_{n-i} \right) \circ d \\
&= \left(\sum_{i=1}^{n-1} t_i \cup t_{n-i} + t_{n-1}d \right) \circ d \\
&= -dt_n d,
\end{aligned}$$

as required. The case $n = 1$ is entirely similar.

For the same reason, to prove (ii) it suffices to show

$$d_n \circ \left(s_n d_{n+1} + \sum_{i=0}^{n} (s_i \cup \tilde{t}_{n+1-i} - t_{n+1-i} \cup s_i) \right) = 0. \tag{3.4}$$

However, applying d on the right side of (3.2) at degree n yields

$$ds_n d = \sum_{i=0}^{n-1} (s_i \cup \tilde{t}_{n-i}) \circ d - \sum_{i=0}^{n-1} (t_{n-i} \cup s_i) \circ d, \tag{3.5}$$

and

$$\begin{aligned}
\sum_{i=0}^{n-1} (s_i \cup \tilde{t}_{n-i}) \circ d &= \sum_{i=0}^{n-1} (s_i \cup \tilde{t}_{n-i}d - s_i d \cup \tilde{t}_{n-i}) \\
&= \sum_{i=0}^{n-1} \Bigg[s_i \cup \left(-d\tilde{t}_{n+1-i} - \sum_{j=1}^{n-i} \tilde{t}_j \cup \tilde{t}_{n+1-i-j} \right) \\
&\qquad - \left(ds_{i+1} - \sum_{j=0}^{i} (s_j \cup \tilde{t}_{i+1-j} - t_{i+1-j} \cup s_j) \right) \cup \tilde{t}_{n-i} \Bigg] \qquad (3.6) \\
&= - \sum_{i=0}^{n} d \circ (s_i \cup \tilde{t}_{n+1-i}) - \sum_{i=0}^{n-1} \sum_{j=0}^{i} t_{i+1-j} \cup s_j \cup \tilde{t}_{n-i}.
\end{aligned}$$

Similarly,

$$- \sum_{i=0}^{n-1} (t_{n-i} \cup s_i) \circ d = \sum_{i=0}^{n} d \circ (t_{n+1-i} \cup s_i) + \sum_{i=0}^{n-1} \sum_{j=0}^{i} t_{i+1-j} \cup s_j \cup \tilde{t}_{n-i}. \tag{3.7}$$

Adding equations (3.5), (3.6), and (3.7), we obtain (3.4). □

The augmentation map $\epsilon\colon \mathfrak{A}_0 \to k$ factors uniquely as $\epsilon = \epsilon_0\epsilon_1$, where $\epsilon_1\colon \mathfrak{A}_0 \to \mathfrak{A}_0/d(\mathfrak{A}_1)$ is the natural map and $\epsilon_0\colon \mathfrak{A}_0/d(\mathfrak{A}_1) \to k$. As a corollary to Proposition 3.2 we obtain the following, which will apply directly to the case of a group extension.

Corollary 3.3. *Let $\mathfrak{A}$ be an augmented DGA over Λ which is exact in positive degrees, and let $\mathfrak{C}$ be a DGCA over Λ such that $\mathfrak{C}_n$ is a projective Λ-module for all $n \geq 0$. Then*

- (i) *Any Λ-homomorphism $t_1\colon \mathfrak{C}_1 \to \mathfrak{A}_0$ satisfying $\epsilon t_1 = 0$ and $\epsilon_1 \circ (t_1 d_2 + t_1 \cup t_1) = 0$ can be extended to a twisting cochain $t\colon \mathfrak{C} \to \mathfrak{A}$.*
- (ii) *If $\tilde{t}$ and t are two twisting cochains such that $\epsilon_1\tilde{t}_1 = \epsilon_1 t_1$, then $\tilde{t}$ and t are homotopic, and there exists a homotopy $s\colon \tilde{t} \simeq t$ such that $s_0 = \eta\epsilon$.*

Proof. Since $\mathrm{Ker}(\epsilon_1) = \mathrm{Im}(d_1)$, the hypothesis of (i) implies $\mathrm{Im}(t_1 d_2 + t_1 \cup t_1) \subseteq \mathrm{Im}(d_1)$. By the projectivity of $\mathfrak{C}_2$, there is a Λ-map $t_2\colon \mathfrak{C}_2 \to \mathfrak{A}_1$ satisfying (3.1) for $i = 2$; (i) now follows from Proposition 3.2(i). To prove (ii), first set $s_0 = \eta\epsilon$, and observe that $\epsilon s_0 = (\epsilon\eta)\epsilon = \epsilon$. Moreover,

$$\epsilon_1 \circ (s_0 d_1 + s_0 \cup \tilde{t}_1 - t_1 \cup s_0) = \epsilon_1\tilde{t}_1 - \epsilon_1 t = 0,$$

as $s_0 d_1 = \eta\epsilon d_1 = 0$. Hence by the projectivity of $\mathfrak{C}_1$, there is a map $s_1\colon \mathfrak{C}_1 \to \mathfrak{A}_1$ satisfying (3.2) for $i = 1$, and (ii) now follows from Proposition 3.2(ii). □

Perturbations. If U is a left DGCM over $\mathfrak{C}$ and V is a left DGM over $\mathfrak{A}$, then there is a chain map

$$\mathrm{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A}) \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_k(U, V), \mathrm{Hom}_k(U, V)),$$

given by $f \mapsto f\cup$. Let $t \in \mathrm{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A})_{-1}$ be a twisting cochain, and set $d_t = d + t\cup$, where d denotes the standard differential on $\mathrm{Hom}_k(U, V)$. We then have

$$d_t^2 = d \circ (t\cup) + (t\cup) \circ d + (t\cup)^2 = d(t\cup) + (t \cup t)\cup = (d(t) + t \cup t)\cup = 0.$$

Hence we may perturb d by $t\cup$ and obtain another differential on $\mathrm{Hom}_k(U, V)$.

There is an analogous construction with tensor products in place of homomorphisms. If M is a left DGM over $\mathfrak{A}$ and N a right DGCM over $\mathfrak{C}$, we define the *cap product map* by:

$$\begin{aligned} \mathrm{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A}) &\longrightarrow \mathrm{Hom}_\Lambda(N \otimes M, N \otimes M) \\ f &\longmapsto f\cap = (1 \times \mu)(1 \times f \times 1)(\Delta \times 1). \end{aligned} \tag{3.8}$$

It follows immediately from Propositions 2.1 and 2.2 that the cap product map is a map of complexes.

Proposition 3.4. *Let M be a left DGM over $\mathfrak{A}$, N a right DGCM over $\mathfrak{C}$.*

- (i) *Let f and g be homogeneous elements of $\mathrm{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A})$. Then $(f\cap) \circ (g\cap) = (f \cup g)\cap$.*
- (ii) *Let $t \in \mathrm{Hom}_\Lambda(\mathfrak{C}, \mathfrak{A})_{-1}$ be a twisting cochain, and set $d_t = d + t\cap\colon N \otimes M \to N \otimes M$. Then $d_t^2 = 0$.*

Proof (cf. [9, Proposition 30.5]). Using the associativity of μ and the coassociativity of Δ, we have

$$\begin{aligned}(f\cap)\circ(g\cap) &= (1\times\mu)(1\times f\times 1)(\Delta\times 1)(1\times\mu)(1\times g\times 1)(\Delta\times 1)\\ &= (1\times\mu)(1\times f\times 1)(1\times 1\times\mu)(\Delta\times 1\times 1)(1\times g\times 1)(\Delta\times 1)\\ &= (1\times\mu)(1\times 1\times\mu)(1\times f\times 1\times 1)(1\times 1\times g\times 1)(\Delta\times 1\times 1)(\Delta\times 1)\\ &= (1\times\mu)(1\times\mu\times 1)(1\times f\times g\times 1)(1\times\Delta\times 1)(\Delta\times 1)\\ &= (1\times\mu)(1\times\mu(f\times g)\Delta\times 1)(\Delta\times 1)\\ &= (1\times\mu)(1\times(f\cup g)\times 1)(\Delta\times 1)\\ &= (f\cup g)\cap,\end{aligned}$$

which proves (i). For (ii), since $d^2=0$, we have

$$d_t^2 = (t\cap)\circ(t\cap)+(t\cap)\circ d + d\circ(t\cap) = (t\cup t)\cap + d(t\cap) = -d(t)\cap + d(t\cap) = 0,$$

as the cap product map is a map of complexes. □

We let $N\otimes_t M$ denote the "twisted tensor product of N and M with respect to t," i.e., the complex with the same graded Λ-module structure as $N\otimes M$ but with differential d_t. Alternatively, we could take M to be a right DGM and N a left DGCM. We could then define a cap product on $M\otimes N$, just as in (3.8), but with

$$f\cap = (\mu\times 1)(1\times f\times 1)(1\times\Delta).$$

We then get the following analogue to Proposition 3.4:

Proposition 3.5. *Let M be a right DGM over $\mathfrak{A}$, N a left DGCM over $\mathfrak{C}$.*

(i) *Let f and g be homogeneous elements of* $\mathrm{Hom}_\Lambda(\mathfrak{C},\mathfrak{A})$. *Then* $(f\cap)\circ(g\cap) = (-1)^{\deg(f)\deg(g)}(g\cup f)\cap$.
(ii) *Let $t\in\mathrm{Hom}_\Lambda(\mathfrak{C},\mathfrak{A})_{-1}$ be a twisting cochain, and set $d_t = d - t\cap\colon M\otimes N\to M\otimes N$. Then $d_t^2=0$.*

4. Special resolutions

Let k be a field, Q a group (not necessarily finite), and $(Y,\partial)\to k$ a free resolution of the trivial kQ-module k. Assume further that $Y_0 = kQ$ and that $\epsilon\colon Y_0\to k$ is the standard augmentation, i.e., $\epsilon(\sigma)=1$ for all $\sigma\in Q$. Let $\Delta\colon Y\to Y\otimes Y$ be a diagonal approximation map (i.e., a kQ-chain map which commutes with the augmentations). Let $\bar Y = k\otimes_{kQ} Y$, let $\pi\colon Y\to\bar Y$ denote the natural map, and let $\bar\partial,\bar\Delta\colon\bar Y\to\bar Y$ and $\bar\epsilon\colon\bar Y_0\to k$ denote the maps induced by ∂, Δ, and ϵ, respectively. Of course, we may identify $\bar Y_0$ with k, and then $\bar\epsilon$ is just the identity map.

Definition 4.1. Let $\imath\in\mathrm{Hom}_k(\bar Y, Y)_0$. We say that $(Y,\partial,\Delta,\imath)$ (or simply Y, when the maps are understood) is a *special resolution* if the following hold:

(i) $\imath_0(1)=1$.
(ii) $\pi\imath = 1_{\bar Y}$.
(iii) For all $n\geq 0$, $\imath(\bar Y_n)$ generates Y_n as a kQ-module.
(iv) $\bar\epsilon\cup 1_{\bar Y} = 1_{\bar Y} = 1_{\bar Y}\cup\bar\epsilon$.
(v) $(1_{\bar Y}\times\bar\Delta)\bar\Delta = (\bar\Delta\times 1_{\bar Y})\bar\Delta$.
(vi) $d(\imath) = \partial_1\imath_1\cup\imath$.

A few remarks will clarify the definition. First, it is always possible to find a map $\imath$ satisfying (i), (ii), and (iii). This can be done, for example, by first choosing a kQ-basis $\{y_n^j\}$ of Y_n, for each $n\geq 0$ (with $\{y_0^j\}=\{1\}$). Then $\{\pi(y_n^j)\}$ is a k-basis for $\bar{Y}_n$, and one can define $\imath(\pi(y_n^j)) = y_n^j$. Conversely, if Y is a special resolution, then these conditions imply that the image under $\imath$ of any k-basis of $\bar{Y}$ is a kQ-basis of Y.

Second, conditions (iv) and (v) are equivalent to demanding that $\bar{Y}$ is a DGCA over k with counit $\bar{\epsilon}$ and comultiplication $\bar{\Delta}$. We remark that (iv) is equivalent to requiring

(iv)$'$ $\imath_0 \cup \imath = \imath = \imath \cup \imath_0$.

(The cup product in (iv)$'$ makes sense, since $\mathrm{Im}(\imath_0) \subseteq Y_0 = kQ$, and Y is a kQ-module.) This follows from the fact that $\bar{\epsilon} \cup 1_{\bar{Y}_n} = \mu\bar{\Delta}_{0,n}$ ($n \geq 0$) and by considering the following commutative diagram:

$$\begin{array}{ccccc} \bar{Y}_n & \xrightarrow{\bar{\Delta}_{0,n}} & \bar{Y}_0 \otimes \bar{Y}_n & \xrightarrow{\imath_0 \times \imath} & Y_0 \otimes Y_n \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ & & \bar{Y}_n & \xrightarrow{\imath} & Y_n \end{array}$$

The interesting condition is (vi). Notice that the cup product here makes sense, since $\mathrm{Im}(\partial_1) \subseteq Y_0 = kQ$. Hence $\partial_1\imath_1 \cup \imath$ in degree n ($n \geq 1$) is the composite

$$\bar{Y}_n \xrightarrow{\bar{\Delta}_{1,n-1}} \bar{Y}_1 \otimes \bar{Y}_{n-1} \xrightarrow{\partial_1\imath_1 \times \imath} Y_0 \otimes Y_{n-1} = kQ \otimes Y_{n-1} \xrightarrow{\mu} Y_{n-1}.$$

Of course, $d(\imath) = \partial\imath - \imath\bar{\partial}$. Hence we may interpret (vi) as an expression for ∂_n in terms of $\bar{\partial}_n$, $\bar{\Delta}_{1,n-1}$ and $\imath$ (and ∂_1).

Example 4.2. The *bar resolution* Y and the standard diagonal approximation map Δ (the "Alexander-Whitney map") are defined by

$$Y_n = \bigoplus_{\sigma_1,\dots,\sigma_n \in Q} kQ(\sigma_1|\cdots|\sigma_n),$$

$$\partial(\sigma_1|\cdots|\sigma_n) = \sigma_1(\sigma_2|\cdots|\sigma_n) + \sum_{i=1}^{n}(-1)^n(\sigma_1|\cdots|\sigma_i\sigma_{i+1}|\cdots|\sigma_n),$$

$$\Delta(\sigma_1|\cdots|\sigma_n) = \sum_{i=0}^{n}(\sigma_1|\cdots|\sigma_i) \otimes \sigma_1\cdots\sigma_i(\sigma_{i+1}|\cdots|\sigma_n).$$

We identify kQ with $Y_0 = kQ()$ by $\sigma \leftrightarrow \sigma()$, and we let $[\sigma_1|\cdots|\sigma_n]$ denote the image of $(\sigma_1|\cdots|\sigma_n)$ in $\bar{Y}$. Define $\imath\colon \bar{Y} \to Y$ by

$$\imath[\sigma_1|\cdots|\sigma_n] = (\sigma_1|\cdots|\sigma_n), \qquad n \geq 0.$$

We claim that $(Y, \partial, \Delta, \imath)$ is a special resolution. It is well known that Δ is coassociative, hence $\bar{\Delta}$ certainly is. All of the other conditions are also easily verified, except for (vi). However,

$$\begin{aligned} d(\imath)[\sigma_1|\cdots|\sigma_n] &= (\partial\imath - \imath\bar{\partial})[\sigma_1|\cdots|\sigma_n] \\ &= (\sigma_1|\cdots|\sigma_n) - \imath\bar{\partial}[\sigma_1|\cdots|\sigma_n] \\ &= (\sigma_1 - 1)(\sigma_2|\cdots|\sigma_n) \\ &= \partial_1(\sigma_1).(\sigma_2|\cdots|\sigma_n) \\ &= (\partial_1\imath_1 \cup \imath)[\sigma_1|\cdots|\sigma_n]. \end{aligned}$$

Example 4.3. The *reduced bar resolution* is the quotient of the bar resolution by the kQ-subcomplex generated by all $(\sigma_1|\cdots|\sigma_n)$ such that $\sigma_i = 1$ for some i $(1 \le i \le n)$. It is easily verified that the reduced bar resolution is special.

Example 4.4. Let Q' and Q'' be groups and set $Q = Q' \times Q''$. Suppose Y' and Y'' are special resolutions for Q' and Q'', respectively. Then $Y = Y' \otimes Y''$ is a projective resolution for the trivial kQ-module k, and we have $Y_0 = kQ' \otimes kQ'' = kQ$. As in Section 2, the diagonal approximations on Y' and Y'' induce a diagonal approximation Δ on Y. Now set $\imath = \imath \times \imath \colon \bar{Y}' \otimes \bar{Y}'' \to Y' \otimes Y''$. We claim that Y is a special resolution. As conditions (i) through (v) are immediate, we check only (vi):

$$\begin{aligned}
\partial_1 \imath_1 \cup \imath &= (\partial_1 \times 1 \ + \ 1 \times \partial_1)(\imath \times \imath)_1 \cup (\imath \times \imath) \\
&= (\partial_1 \imath_1 \times \imath_0) \cup (\imath \times \imath) \ + \ (\imath_0 \times \partial_1 \imath_1) \cup (\imath \times \imath) \\
&= (\partial_1 \imath_1 \cup \imath) \times (\imath_0 \cup \imath) \ + \ (\imath_0 \cup \imath) \times (\partial_1 \imath_1 \cup \imath) \\
&= (\partial_1 \imath_1 \cup \imath) \times \imath \ + \ \imath \times (\partial_1 \imath_1 \cup \imath) \\
&= d(\imath) \times \imath \ + \ \imath \times d(\imath) \\
&= d(\imath \times \imath) \\
&= d(\imath).
\end{aligned}$$

Example 4.5. Let Q be an elementary abelian 2-group and k a field of characteristic 2. Then the minimal resolution of the trivial kQ-module is special. This follows from the two preceding examples, since the reduced bar resolution of a cyclic group of order 2 is minimal, and the tensor product of minimal resolutions is minimal.

Suppose, however, that p is an odd prime, k is a field of characteristic p, $Q = \langle \sigma \mid \sigma^p = 1 \rangle$, and (Y, ∂) is the minimal resolution. Then for no choice of Δ and $\imath$ is Y special. For in this case we have $\bar{Y} = H_*(Q,k)$, $\bar{\partial} = 0$, and $\bar{\Delta}$ is the comultiplication in homology. Since p is odd, $\bar{\Delta}_{1,1} = 0$, so by (vi),

$$\partial_2 \imath_2 = \partial_1 \imath_1 \cup \imath_1 = \mu(\partial_1 \times \imath_1)\bar{\Delta}_{1,1} = 0.$$

Hence $\imath_2(\bar{Y}_2) \subseteq \mathrm{Ker}(\partial_2) = \mathrm{rad}_{kQ}(Y_2)$, contradicting (iii).

We conclude with a technical fact concerning special resolutions which will be used in the case of a group extension.

Proposition 4.6. *Let $(Y, \partial, \Delta, \imath)$ be a special resolution for the trivial kQ-module k. Set $\phi = \partial_1 \imath_1 \colon \bar{Y}_1 \to kQ$. Then $\phi \bar{\partial}_2 = \phi \cup \phi$.*

Proof. Start with condition (vi) of the definition at degree 2, and apply ∂_1 to obtain

$$\begin{aligned}
\partial_2 \imath_2 - \imath_1 \bar{\partial}_2 &= \partial_1 \imath_1 \cup \imath_1 \\
-\partial_1 \imath_1 \bar{\partial}_2 &= -\partial_1 \imath_1 \cup \partial_1 \imath_1,
\end{aligned}$$

as claimed. □

5. The twisting cochain of a group extension

Let k be a field and let $1 \to H \to G \to Q \to 1$ be an extension of (not necessarily finite) groups. In this section, we show how to associate to the group extension an augmented DGA $\mathfrak{A}$ over k, a DGCA $\mathfrak{C}$, and a twisting cochain $t \in \mathrm{Hom}_k(\mathfrak{C}, \mathfrak{A})_{-1}$.

Recall that for a kH-module U, $U^G = kG \otimes_{kH} U$ is a kG-module (the induced module). For $g \in G$, we let U^g denote the kH-module with the same underlying vector space structure as U, but with a new action given by $h.x = ghg^{-1}x$, for $h \in H$ and $x \in U$. For a kG-module V, we let V_H denote V considered as a kH-module (the restriction of V to H; cf. Alperin [1]).

Let $(Y, \partial, \Delta, \imath)$ be a special resolution for the trivial kQ-module. We let $\mathfrak{C}$ be the DGCA $\bar{Y} = k \otimes_{kQ} Y$, with differential $d = \bar{\partial}$ and comultiplication $\bar{\Delta}$.

Let $(P, \partial) \xrightarrow{\epsilon} k$ be a projective resolution for the trivial kH-module, and set

$$\mathfrak{A} = \hom_{kG}(P^G, P^G)^{\mathrm{op}}. \tag{5.1}$$

Thus $\mathfrak{A}$ is a DGA over k with multiplication $\mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\mu} \mathfrak{A}$ given by right composition (which involves a sign). To define the augmentation $\epsilon \colon \mathfrak{A}_0 \to k$ we first define a map $\epsilon_1 \colon \mathfrak{A}_0 \to kQ$, as follows. Since any $f \in \mathfrak{A}_0$ is a map of complexes, there exists a unique $\alpha \in kQ$ such that right multiplication by α $(\cdot\alpha)$ makes the following diagram commute:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_1^G & \longrightarrow & P_0^G & \xrightarrow{\epsilon^G} & k^G = kQ \\
 & & \Big\downarrow f_1 & & \Big\downarrow f_0 & & \Big\downarrow \cdot\alpha \\
\cdots & \longrightarrow & P_1^G & \longrightarrow & P_0^G & \xrightarrow{\epsilon^G} & k^G = kQ.
\end{array}$$

We define $\epsilon_1(f) = \alpha$, and we set $\epsilon = \epsilon_0\epsilon_1$, where $\epsilon_0 \colon kQ \to k$ is the standard augmentation. It follows that

$$\epsilon_1\mu(f \otimes g) = \epsilon_1(gf) = \epsilon_1(f)\epsilon_1(g),$$

hence ϵ_1 is a map of algebras (which was the reason for choosing the *opposite* endomorphism algebra in (5.1)). Moreover, for $f \in \mathfrak{A}_1$, $d_1(f) = \partial f + f\partial$, which in degree 0 is just $\partial_1 f_0$, so $\epsilon_1 d_1(f) = 0$. We conclude that ϵ_1 is a map of DGAs, and since ϵ_0 is clearly a map of DGAs, so is $\epsilon = \epsilon_0\epsilon_1$.

Since $P^G \to k^G = kQ$ is just a projective resolution of the kG-module kQ, the Comparison Theorem implies that the sequence

$$\cdots \to \mathfrak{A}_2 \to \mathfrak{A}_1 \to \mathfrak{A}_0 \xrightarrow{\epsilon_1} kQ \to 0,$$

is exact. Hence $\mathfrak{A}$ is exact in positive degrees, we may identify $\mathfrak{A}_0/\operatorname{Im}(d_1)$ with kQ, and the factorization $\epsilon = \epsilon_0\epsilon_1$ agrees with the notation of Section 3.

We now turn to the definition of t. Let $\phi = \partial_1\imath_1 \colon \mathfrak{C}_1 \to Y_0 = kQ$. Since $\mathfrak{C}_1$ is k-projective, there exists a k-linear map $t_1 \colon \mathfrak{C}_1 \to \mathfrak{A}_0$ such that $\epsilon_1 t_1 + \phi = 0$. By Proposition 4.6,

$$\begin{aligned}
\epsilon_1 \circ (t_1 \cup t_1 + t_1 d_2) &= \epsilon_1 t_1 \cup \epsilon_1 t_1 + \epsilon_1 t_1 d_2 \\
&= \phi \cup \phi - \phi d_2 \\
&= 0.
\end{aligned}$$

Hence we can apply Corollary 3.3 to immediately prove

Theorem 5.1. *Let $1 \to H \to G \to Q \to 1$ be a group extension, k a field, $Y \to k$ a special kQ-resolution, and $P \to k$ an arbitrary kH-projective resolution. Let $\mathfrak{C} = k \otimes_{kQ} Y$ and $\mathfrak{A} = \hom_{kG}(P^G, P^G)^{\mathrm{op}}$, and let ϵ_1 and ϕ be defined as above. Then there exists a twisting cochain $t \colon \mathfrak{C} \to \mathfrak{A}$ with $\epsilon_1 t_1 + \phi = 0$. Moreover, any two such twisting cochains are homotopic, and the homotopy can be chosen to be $\eta\epsilon$ in degree 0.*

We will call a map t satisfying the conditions of Theorem 5.1 a *twisting cochain associated to the group extension* $1 \to H \to G \to Q \to 1$.

6. Projective resolutions

We continue with the notation of Section 5: k is a field, $1 \to H \to G \to Q \to 1$ is a group extension, P is a kH-projective resolution, Y is a special kQ-resolution, and $\mathfrak{A}$, $\mathfrak{C}$ and t are defined as before. We now show how the twisting cochain t relates to a construction of C. T. C. Wall in [12]. In that paper, Wall showed that for any free resolution Y, a differential could be imposed on $P^G \otimes \bar{Y}$ to form a projective resolution of the trivial kG-module, and he provided an inductive method for finding such a differential. Here, we show—with our additional assumption that Y is special—that this differential can be constructed as the "twisted differential" d_t arising from our twisting cochain t.

By Proposition 2.1(iii), right evaluation (with a sign) $P^G \otimes \mathfrak{A} \xrightarrow{\mu} P^G$ is a chain map, and therefore gives P^G the structure of a right DGM over $\mathfrak{A}$. Since $\bar{Y}$ is a left DGCM over itself, we can apply Proposition 3.5(ii) to form the twisted tensor product $X = P^G \otimes_t \bar{Y}$. We define $\epsilon_X : X \to k$ as the composite

$$P^G \otimes_t \bar{Y} \xrightarrow{\epsilon^G \times 1} kQ \otimes \bar{Y} \xrightarrow{\epsilon_0 \times \bar{\epsilon}} k \otimes k \xrightarrow{\mu} k.$$

Now P^G, $\mathfrak{C} = \bar{Y}$, and $\mathfrak{A}$ are all kG-modules, with G acting trivially on $\mathfrak{C}$ and $\mathfrak{A}$. In particular, each X_n is a kG-module—in fact, X_n is a projective kG-module, as it is isomorphic to the direct sum of modules of the form P_i^G. Moreover, ϵ_X, as a composite of kG-homomorphisms, is a kG-homomorphism. Finally, the differential $d_t = d - t\cap$ is a kG-homomorphism. This follows from the observation that $P^G \otimes \mathfrak{A} \xrightarrow{\mu} P^G$ preserves the action of G (and t trivially preserves the G-action), so one can apply the perturbation construction of Section 3 with $\Lambda = kG$.

Theorem 6.1. *Let k be a field, $1 \to H \to G \to Q \to 1$ a group extension, $P \xrightarrow{\epsilon} k$ a kH-projective resolution, $Y \xrightarrow{\epsilon} k$ a special kQ-resolution, and t an associated twisting cochain. Then $P^G \otimes_t \bar{Y} \xrightarrow{\epsilon_X} k$ is a projective resolution of the trivial kG-module.*

Proof. We first check that $\epsilon_X \circ (d_t)_1 = 0$. For this it suffices to show $\epsilon_X \circ (t_1\cap) = 0$. But

$$\begin{aligned} (\epsilon^G \times 1) \circ (t_1\cap) &= (\epsilon^G \times 1)(\mu \times 1)(1 \times t_1 \times 1)(1 \times \bar{\Delta}) \\ &= (\mu \times 1)(\epsilon^G \times \epsilon_1 \times 1)(1 \times t_1 \times 1)(1 \times \bar{\Delta}) \\ &= -(\mu \times 1)(\epsilon^G \times \phi \times 1)(1 \times \bar{\Delta}) \end{aligned} \tag{6.1}$$

Hence

$$\begin{aligned} \epsilon_X \circ (t_1\cap) &= \mu(\epsilon_0 \times \bar{\epsilon})(\epsilon^G \times 1)(t_1\cap) \\ &= -\mu(\epsilon_0 \times \bar{\epsilon})(\mu \times 1)(\epsilon^G \times \phi \times 1)(1 \times \bar{\Delta}) \\ &= \mu(1 \times \bar{\epsilon})(\mu \times 1)(\epsilon_0\epsilon^G \times \epsilon\phi \times 1)(1 \times \bar{\Delta}) \\ &= 0, \end{aligned}$$

as $\epsilon\phi = (\epsilon\partial_1)\imath_1 = 0$.

We next show $H_*(X, d_t) = k$, completing the proof. To see this, consider the spectral sequence arising from the bigraded complex defined by

$$A^{p,q} = P^G_{-q} \otimes \bar{Y}_{-p}, \qquad \delta = d_t = d - t\cap.$$

Writing $\delta = \delta_0 + \delta_1 + \cdots$, as in Section 2, we see that

$$\delta_r(x \otimes \bar{y}) = \begin{cases} \partial(x) \otimes \bar{y} & \text{if } r = 0, \\ (-1)^q x \otimes \bar{\partial}(\bar{y}) - t_1 \cap (x \otimes \bar{y}) & \text{if } r = 1, \\ -t_r \cap (x \otimes \bar{y}) & \text{if } r \geq 2, \end{cases}$$

for $x \in P^G_{-q}$, $\bar{y} \in \bar{Y}_{-p}$. Each column is a direct sum of copies of P^G, and $d_0 = \delta_0$ induces the standard differential on each copy, hence

$$E_1^{p,q} \cong \begin{cases} kQ \otimes \bar{Y}_{-p} & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

The isomorphism is given as follows: $\alpha \otimes \bar{y} \in kQ \otimes \bar{Y}_{-p}$ is represented in $A^{p,q}$ by $x \otimes \bar{y}$, where x is any element of P_0^G such that $\epsilon^G(x) = \alpha$.

Consider the kQ-homomorphism $\mu \circ (1 \times i) \colon \bar{Y}^Q = kQ \otimes \bar{Y} \to Y$. By Definition 4.1(iii), this map is surjective, and by comparing dimensions, we see that it is an isomorphism. Using this isomorphism to identify E_1 with Y, we claim that d_1 induces the standard differential ∂ on Y. This will complete the proof, since we will then have $H_*(X, d_t) = E_\infty = E_2 = k$ (concentrated in degree 0). But since d_1 is a kQ-homomorphism, it suffices to check this for $y = \imath(\bar{y})$, for any $\bar{y} \in \bar{Y}$. Then y corresponds to $\zeta \in E_1^{p,0}$, where ζ is represented by $x \otimes \bar{y}$, where $x \in P_0^G$ and $\epsilon^G(x) = 1$. But by Definition 4.1(vi) and (6.1),

$$\begin{aligned} -\mu(1 \times \imath)(\epsilon^G \times 1)(t_1 \cap (x \times \bar{y})) &= \mu(1 \times \imath)(\mu \times 1)(\epsilon^G \times \phi \times 1)(1 \times \bar{\Delta})(x \times \bar{y}) \\ &= \mu(1 \times \imath)(\phi \times 1)\bar{\Delta}(\bar{y}) \\ &= (\phi \cup \imath)(\bar{y}) \\ &= d(\imath)(\bar{y}) \\ &= \partial\imath(\bar{y}) - \imath\bar{\partial}(\bar{y}). \end{aligned}$$

Hence $d_1(\zeta)$ corresponds to

$$\begin{aligned} \mu(1 \times \imath)(\epsilon^G \times 1)\delta_1(x \otimes \bar{y}) &= \mu(1 \times \imath)(\epsilon^G \times 1)(x \otimes \bar{\partial}(\bar{y}) - t_1 \cap (x \otimes \bar{y})) \\ &= \imath\bar{\partial}(\bar{y}) + \partial\imath(\bar{y}) - \imath\bar{\partial}(\bar{y}) \\ &= \partial(y), \end{aligned}$$

as claimed. □

7. The explicit twisting cochain of a split extension of finite groups

We now concentrate on the case where $G = H \rtimes Q$ is a *split* extension of *finite* groups. In this case another description of $\mathfrak{A}$ is useful. Consider the augmented DGA

$$\tilde{\mathfrak{A}} = kQ \otimes \hom_k(P, P)^{\mathrm{op}}.$$

As usual, kQ is considered to be an augmented DGA concentrated in degree 0 with the usual augmentation map ϵ_0. We define $\hat{\mathfrak{A}}$ to be the k-submodule of $\tilde{\mathfrak{A}}$ generated by all $\sigma \otimes \phi$, where $\sigma \in Q$ and $\phi \in \hom_{kH}(P, P^{\sigma^{-1}})$. Notice that $\hat{\mathfrak{A}}$ contains the unit element, is closed under multiplication and the differential, and hence is an augmented DGA. As complexes,

$$\begin{aligned}\hat{\mathfrak{A}} &= \bigoplus_{\sigma\in Q} \sigma\otimes \hom_{kH}(P, P^{\sigma^{-1}}) \cong \hom_{kH}\left(\bigoplus_{\sigma\in Q}\sigma\otimes P, P\right)\\ &= \hom_{kH}((P^G)_H, P) \cong \hom_{kG}(P^G, P^G) = \mathfrak{A}.\end{aligned}$$

The isomorphism $\hat{\mathfrak{A}} \to \mathfrak{A}$ is given as follows: $\sigma\otimes f \in \hat{\mathfrak{A}}$ corresponds to the map $\gamma \in \mathfrak{A}$ defined by $\gamma(1\otimes x) = \sigma\otimes f(x)$. One may check that this isomorphism preserves the multiplication and commutes with the augmentations, hence $\hat{\mathfrak{A}}$ and $\mathfrak{A}$ are isomorphic as augmented DGAs.

We next show how to define an explicit twisting cochain for the extension. Because H is finite, we can take P to be the *minimal* resolution. This resolution is characterized by the following property: if $X \to k$ is any projective resolution of the trivial kH-module, then there exist kH-chain maps $\iota\colon P \to X$ and $\pi\colon X \to P$ commuting with the augmentations such that $\pi\iota$ is the identity on P. In particular, we can take X to be any complex of kG-modules which becomes a kH-projective resolution upon restriction to H (for example, a kG-projective resolution of the trivial module). Fix such a choice of X, ι, and π for the remainder of this section. We then have

Proposition 7.1. *There exists a map $s \in \mathrm{Hom}_{kH}(X, X)_1$ such that $s\iota = 0$, $\pi s = 0$, $s^2 = 0$, and $\partial s + s\partial = 1 - \iota\pi$.*

Proof. Let $W = \mathrm{Ker}(\pi)$, so $X = \iota(P) \oplus W$ as kH-complexes. W is a projective resolution of 0, and is therefore contractible. Hence if we let $V_n = \mathrm{Ker}(\partial_n)$ $(n \in \mathbb{Z})$, then there is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & 0\\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ \cdots & \longrightarrow & V_2\oplus V_1 & \longrightarrow & V_1\oplus V_0 & \longrightarrow & V_0\oplus 0 & \longrightarrow & 0\end{array}$$

where the maps along the bottom row are given by $(v_n, v_{n-1}) \mapsto (v_{n-1}, 0)$ $(v_i \in V_i)$. Define $s' \in \mathrm{Hom}_{kH}(W, W)_1$ by $s'(v_n, v_{n-1}) = (0, v_n)$. It is clear that $(s')^2 = 0$ and $\partial s' + s'\partial$ is the identity on W. Extend s' to a map $s \in \mathrm{Hom}_{kH}(X, X)_1$ by $s(x + y) = s'(y)$ $(x \in \iota(P),\ y \in W)$. ☐

Fix a map s as in Proposition 7.1, let $\phi = \partial_1\imath_1$, and define $t \in \mathrm{Hom}_k(\mathfrak{C}, \mathfrak{A})_{-1}$ by letting t_n $(n \geq 1)$ be $(-1)^n$ times the composite

$$\mathfrak{C}_n = \bar{Y}_n \xrightarrow{\bar{\Delta}_{1^n}} \bar{Y}_1^{\otimes n} \xrightarrow{\phi\times\cdots\times\phi} kQ^{\otimes n} \xrightarrow{w_n} \hat{\mathfrak{A}}_{n-1} \cong \mathfrak{A}_{n-1},$$

where $w_n(\sigma_1\otimes\sigma_2\otimes\sigma_3\otimes\cdots\otimes\sigma_n) = \sigma_1\sigma_2\sigma_3\cdots\sigma_n \otimes \pi\sigma_n^{-1}s\cdots\sigma_3^{-1}s\sigma_2^{-1}s\sigma_1^{-1}\iota$.

Theorem 7.2. *Let $G = H \rtimes Q$ be a split extension of finite groups, k a field, $P \to k$ the minimal kH-projective resolution, $Y \to k$ a special kQ-resolution, and define a map t as above. Then t is a twisting cochain associated to the extension.*

For proving the theorem, it is useful to have another description of t. Define

$$\begin{aligned}
&\hat\iota \in \operatorname{Hom}_k(\bar Y_0, \hom_{kG}(P^G, (X_H)^G)_0) && \text{by} && \hat\iota(1) = \iota^G,\\
&\hat\pi \in \operatorname{Hom}_k(\bar Y_0, \hom_{kG}((X_H)^G, P^G)_0) && \text{by} && \hat\pi(1) = \pi^G,\\
&\hat s \in \operatorname{Hom}_k(\bar Y_0, \hom_{kG}((X_H)^G, (X_H)^G)_1^{\mathrm{op}}) && \text{by} && \hat s(1) = s^G,\\
&\hat\phi \in \operatorname{Hom}_k(\bar Y_1, \hom_{kG}((X_H)^G, (X_H)^G)_0^{\mathrm{op}}) && \text{by} && \hat\phi = \beta\circ\phi,
\end{aligned}$$

where $\beta\colon kQ \to \hom_{kG}((X_H)^G, (X_H)^G)_0^{\mathrm{op}}$ is the morphism of DGAs defined by $\beta(\sigma)(1\otimes x) = \sigma\otimes\sigma^{-1}x$ ($\sigma\in Q$, $x\in X$). Then we may form cup products using right compositions for multiplications, and we see that

$$t_n = -\hat\iota \cup (\hat\phi\cup\hat s)^{\cup(n-1)} \cup \hat\phi \cup \hat\pi, \qquad n\geq 1. \tag{7.1}$$

Notice that for (7.1) to hold requires not only the coassociativity of $\bar\Delta$, but part (iv) of Definition 4.1 as well.

Proof of Theorem 7.2. For notational convenience, we set $a = \hat\iota$, $b = \hat\phi$, $c = \hat s$, $z = \hat\pi$, and suppress the cup product symbol. Let $x = bc$, so

$$t = -\sum_{n\geq 0} ax^n bz.$$

We have $d(a) = 0 = d(z)$, as ι and π are chain maps, while Proposition 4.6 implies $d(b) = b^2$, and Proposition 7.1 implies $d(c) = 1 - za$. Now,

$$d(x) = d(b)c - bd(c) = bx - b + bza,$$

whence

$$\begin{aligned}
\sum_{n\geq 0} d(x^n) &= \sum_{0\leq i\leq n-1} x^i d(x) x^{n-i-1}\\
&= \sum x^i b x^{n-i} - \sum x^i b x^{n-i-1} + \sum x^i bza x^{n-i-1}\\
&= -\sum_{n\geq 0} x^n b + \sum_{0\leq i\leq n-1} x^i bza x^{n-i-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
d(t) &= -\sum_{n\geq 0} a d(x^n) bz - \sum_{n\geq 0} ax^n d(b) z\\
&= \sum_{n\geq 0} ax^n b^2 z + \sum_{0\leq i\leq n-1} ax^i bzax^{n-i-1} bz - \sum_{n\geq 0} ax^n b^2 z\\
&= -t^2.
\end{aligned}$$

Finally, $\epsilon_1 t_1 = -\epsilon_1 w_1 \phi = -\phi$, so t is a twisting cochain associated to the group extension. □

8. The LHS spectral sequence of a split extension of finite groups

We continue to examine the split extension of finite groups $G = H\rtimes Q$. As before, let Y be a special kQ-resolution, P the minimal kH-resolution, X a kG-resolution, and t an arbitrary associated twisting cochain. Let M be a kG-module such that M_H is semisimple, i.e. $\operatorname{rad}_{kH}(M) = 0$. Construct the LHS spectral sequence of the extension with coefficients in M from the double complex

$$E_0 = \operatorname{Hom}_{kQ}(Y, \operatorname{Hom}_{kH}(X, M)). \tag{8.1}$$

In accordance with the standard sign conventions, the differentials $d'\colon E_0^{p,q} \to E_0^{p+1,q}$ and $d''\colon E_0^{p,q} \to E_0^{p,q+1}$ are given by

$$d'(f)(y) = (-1)^{p+q+1} f(\partial(y)) = -\partial^*(f)(y),$$
$$d''(f)(y) = (-1)^{q+1} f(y) \circ \partial = -\partial^*(f(y)),$$

for $f \in E_0^{p,q}$ and $y \in Y_p$. We have

$$E_1 \cong \mathrm{Hom}_{kQ}(Y, H^*(H,M)), \tag{8.2}$$

and since P is the minimal resolution and M_H is semisimple,

$$H^*(H,M) = \mathrm{Hom}_{kH}(P,M) \cong \mathrm{Hom}_{kG}(P^G, M).$$

At the same time we consider the spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ arising from the bigraded complex $A = \mathrm{Hom}_k(\bar{Y}, H^*(H,M))$ endowed with the differential δ, which is defined as follows: Composition

$$\mathrm{hom}_{kG}(P^G, P^G)^{\mathrm{op}} \otimes \mathrm{Hom}_{kG}(P^G, M) \to \mathrm{Hom}_{kG}(P^G, M)$$

gives $H^*(H,M)$ the structure of a left DGM over $\mathfrak{A}$, hence there is a cup product $\mathrm{Hom}_k(\mathfrak{C}, \mathfrak{A}) \otimes A \xrightarrow{\cup} A$ (and this makes A a left DGM over $\mathrm{Hom}_k(\mathfrak{C}, \mathfrak{A})$). We let $\delta = d + t\cup$, where d is the usual differential on A. As we saw in Section 3, $\delta^2 = 0$.

Since the differential in $H^*(H,M)$ is 0, we have

$$\delta_r(f) = \begin{cases} 0 & \text{if } r = 0, \\ (-1)^{p+q+1} f d + t_1 \cup f & \text{if } r = 1, \\ t_r \cup f & \text{if } r \geq 2, \end{cases}$$

for $f \in A^{p,q}$. For the same reason, we have

$$(dg) \cup f = d \circ (g \cup f) = 0,$$

for all $g \in \mathrm{Hom}_k(\mathfrak{C}, \mathfrak{A})$ and $f \in A$ (keep in mind $dg = d \circ g$ in our notation). Since $\delta_0 = 0$, $\tilde{E}_1 = A$. Moreover, the map $\Phi\colon A \to E_1$, defined by

$$\Phi(f)(\imath(\bar{y})) = f(\bar{y}), \qquad \bar{y} \in \bar{Y},\ f \in A, \tag{8.3}$$

is an isomorphism. We can now state our main theorem:

Theorem 8.1. *Let $G = H \rtimes Q$ be a split extension of finite groups, k a field, and M a kG-module which is semisimple as a kH-module. Let $Y \to k$ be a special kQ-resolution, $P \to k$ the minimal kH-resolution, and construct the LHS spectral sequence (E_r, d_r) as above. Let t be a twisting cochain associated to the extension, and construct the spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ using t as above. Then the map $\Phi\colon \tilde{E}_1 \to E_1$ induces an isomorphism $\Phi_r\colon \tilde{E}_r \to E_r$ for all $r \geq 1$.*

To prove the theorem, we first show that the spectral sequence $\tilde{E}$ is independent of the choice of twisting cochain t. This leaves us free to use the explicit twisting cochain defined in Section 7.

Lemma 8.2. *Suppose $\hat{t}$ is another twisting cochain associated to the group extension, and let $(\hat{E}_r, \hat{d}_r)$ denote the spectral sequence arising from the bigraded complex A endowed with the differential $\hat{\delta} = d + \hat{t}\cup$. Then $\hat{d}_r = \tilde{d}_r$ for all $r \geq 1$.*

Proof. By Corollary 3.3, there is a homotopy $s\colon \hat{t} \simeq t$ with $s_0 = \eta\epsilon$. For $r = 1$, equation (3.2) yields

$$d_1 s_1 = s_0 \cup \hat{t}_1 - t_1 \cup s_0 = \hat{t}_1 - t_1,$$

for $f \in A^{p,q}$, so that indeed

$$\hat{\delta}_1(f) - \delta_1(f) = (\hat{t}_1 - t_1) \cup f = d_1 s_1 \cup f = 0.$$

Now assume $r \geq 2$ and $\hat{d}_n = \tilde{d}_n$ for $1 \leq n < r$. Let $\zeta \in \tilde{E}_r^{p,q}$ and choose $\hat{f} \in F^p A^{p+q}$ such that $\zeta = \pi^{p,q}(\hat{f}) + \tilde{B}_r^{p,q}$ and $\hat{\delta}(\hat{f}) \in F^{p+r} A^{p+q+1}$. By definition,

$$\hat{d}_r(\zeta) = \pi^{p+r,q-r+1}\hat{\delta}(\hat{f}) + \tilde{B}_r^{p+r,q-r+1}.$$

Let $f = s \cup \hat{f}$. Since $s_0 = \eta\epsilon$, $\pi^{p,q}(\hat{f}) = \pi^{p,q}(f)$. Moreover,

$$\begin{aligned}
\delta(f) &= d(s \cup \hat{f}) + t \cup s \cup \hat{f} \\
&= d(s) \cup \hat{f} + s \cup d(\hat{f}) + (s \cup \hat{t} - d(s)) \cup \hat{f} \\
&= s \cup (d(\hat{f}) + \hat{t} \cup \hat{f}) \\
&= s \cup \hat{\delta}(\hat{f}).
\end{aligned}$$

Hence $\pi^{p+r,q-r+1}\hat{\delta}(\hat{f}) = \pi^{p+r,q-r+1}\delta(f)$, and therefore $\hat{d}_r(\zeta) = \tilde{d}_r(\zeta)$. □

Lemma 8.3. *There exist k-linear maps $\eta\colon E_1 \to E_0$ of bidegree $(0,0)$, $\rho\colon E_0 \to E_1$ of bidegree $(0,0)$, and $T\colon E_0 \to E_0$ of bidegree $(1,-1)$ such that*

(i) *$\eta(E_1) \subseteq Z_1$, $\rho|_{Z_1}$ is the natural map, and $\rho\eta = 1_{E_1}$.*
(ii) *$d'(f) + d''T(f) = 0$, for all $f \in E_0$ such that $d'(f) \in \operatorname{Im}(d'')$.*
(iii) *$\delta_r = \Phi^{-1}\rho d' T^{r-1}\eta\Phi$, for all $r \geq 1$.*

Proof. Let $a = \hat{\imath}\cup$, $b = \hat{\phi}\cup$, $c = \hat{s}\cup$, and $z = \hat{\pi}\cup$. For any kQ-module U, let $R = \imath^*\colon \operatorname{Hom}_{kQ}(Y, U) \to \operatorname{Hom}_k(\bar{Y}, U)$ denote precomposition with $\imath$. It follows from Definition 4.1 that R is an isomorphism. Hence we may set $\eta = R^{-1}zR$ and $\rho = R^{-1}aR$, and then (i) is easily verified. To define T, begin with the observation that if $\gamma \in B\operatorname{Hom}_{kH}(X, M)$, then γ represents the zero element of $H^*(H, M)$, so $\gamma\iota = 0$. Therefore

$$\gamma = \gamma(1 - \iota\pi) = \gamma(\partial s + s\partial) = (\gamma s)\partial.$$

(So s^* is a section of the map $\partial\colon \operatorname{Hom}_{kH}(X, M) \to B\operatorname{Hom}_{kH}(X, M)$.) Hence we let $T = -R^{-1}cRd'$. Then if $d'(f) \in \operatorname{Im}(d'') = \operatorname{Hom}_{kQ}(Y_{p+1}, B_{-q}\operatorname{Hom}_{kH}(X, M))$, we have

$$(d''T(f))(y) = (-1)^q T(f)(y) \circ \partial = -d'(f)(y) \circ s\partial = -d'(f)(y)$$

for all $y \in \iota(\bar{Y})$, proving (ii).

We now turn to the proof of (iii). By Definition 4.1(vi), we have

$$-Rd' = \imath^*\partial^* = (\partial\imath)^* = (\imath\bar{\partial} + \phi \cup \imath)^* = \bar{\partial}^* R + bR,$$

Furthermore, $\bar{\partial}^*$ commutes with a, c, and z, and $az = 1$. Hence

$$\Phi^{-1}\rho d'\eta\Phi = RR^{-1}a(Rd')R^{-1}zRR^{-1} = -a(\bar{\partial}^* + b)z = -\bar{\partial}^* - abz = d + t_1\cup,$$

and we have verified (iii) for $r = 1$. Now, Proposition 7.1 implies $ac = c^2 = cz = 0$, so for $r \geq 2$,

$$\begin{aligned}\Phi^{-1}\rho d' T^{r-1}\eta\Phi &= aRd'(-R^{-1}cRd')^{r-1}R^{-1}z\\ &= -a(\bar{\partial}^* + b)c(\bar{\partial}^* + b)\cdots c(\bar{\partial}^* + b)z\\ &= -a(bc)^{r-1}bz\\ &= t_r\cup,\end{aligned}$$

which proves (iii). □

Proof of Theorem 8.1. The proof is by induction on r. By Lemma 8.3(iii), $\Phi\delta_1 = \rho d'\eta\Phi = d_1\Phi$, so the theorem holds for $r = 1$.

Let $\pi_{i,j}\colon Z_i/B_j \to E_i$ denote the natural map ($j \leq i$). Fix $r \geq 2$, and assume $\Phi_s\tilde{d}_s = d_s\Phi_s$ for all $1 \leq s < r$, where $\Phi_i\colon \tilde{E}_i \to E_i$ satisfies $\Phi_i(\tilde{x} + \tilde{B}_i) = \pi_{i,1}\Phi(\tilde{x})$ ($\tilde{x} \in \tilde{Z}_i$, $1 \leq i \leq r$). We must show $\Phi_r\tilde{d}_r = d_r\Phi_r$.

Let $\tilde{\zeta} \in \tilde{E}_r^{p,q}$, $\zeta = \Phi_r(\tilde{\zeta})$. By definition, there are $\tilde{x}_i \in \tilde{E}_0^{p+i,q-i}$ ($0 \leq i \leq r-2$) with $\tilde{x}_0 \in \tilde{Z}_r^{p,q}$, $\tilde{\zeta} = \tilde{x}_0 + \tilde{B}_r^{p,q}$,

$$\sum_{j=0}^{i}\delta_{i-j+1}(\tilde{x}_j) = 0, \qquad 0 \leq i \leq r-2, \tag{8.4}$$

and

$$\tilde{d}_r(\tilde{\zeta}) = \sum_{i=0}^{r-2}\delta_{r-i}(\tilde{x}_i) + \tilde{B}_r^{p+r,q-r+1}.$$

We next produce $z_j \in E_0^{p+j,q-j}$ ($0 \leq j \leq r-1$) satisfying $\zeta_r = \pi_{r,0}(z_0)$ and

$$d'(z_i) + d''(z_{i+1}) = 0, \qquad 0 \leq i \leq r-2. \tag{8.5}$$

It will then follow that $d_r(\zeta) = \pi_{r,0}(d'(z_{r-1}))$. To do this, let $x_i = \Phi(\tilde{x}_i)$,

$$z_i = \sum_{j=0}^{i} T^{i-j}\eta(x_j), \qquad 0 \leq i \leq r-2,$$

and $z_{r-1} = T(z_{r-2})$. We prove (8.5) by induction on i. For $i = 0$, we are to show

$$d'\eta(x_0) + d''T\eta(x_0) + d''\eta(x_1) = 0. \tag{8.6}$$

But by Lemma 8.3(i), $d''\eta = 0$, and since $\eta(x_0) \in Z_r \subseteq Z_1$, equation (8.6) follows from Lemma 8.3(ii).

Now suppose $1 \leq i \leq r-3$ and $d'(z_{i-1}) + d''(z_i) = 0$. Then $d''d'(z_i) = -d'd''(z_i) = 0$, so $d'(z_i) \in \mathrm{Ker}(d'') = Z_1$. But

$$\rho d'(z_i) = \sum_{j=0}^{i}\rho d' T^{i-j}\eta(x_j) = \sum_{j=0}^{i}\Phi\delta_{i-j+1}(\tilde{x}_j) = 0,$$

by Lemma 8.3(iii) and (8.4). Since $\rho|_{Z_1}$ is the natural map, $d'(z_i) \in \mathrm{Im}(d'')$. Now since

$$d''T(z_i) = d''\left(\sum_{j=0}^{i} T^{i-j+1}\eta(x_j)\right) = d''\left(\sum_{j=0}^{i} T^{i-j+1}\eta(x_j) + \eta(x_{i+1})\right) = d''(z_{i+1}),$$

Lemma 8.3(ii) implies

$$0 = d'(z_i) + d''T(z_i) = d'(z_i) + d''(z_{i+1}),$$

which completes the inductive step. Finally, $d'(z_{r-3}) + d''(z_{r-2}) = 0$, so as above one obtains $d'(z_{r-2}) \in \operatorname{Im}(d'')$, and in the same way completes the proof of (8.5).

Finally,

$$\begin{aligned} d_r(\zeta) &= \pi_{r,0} d'(z_{r-1}) = \pi_{r,0} \sum_{j=0}^{r-2} d'T^{r-j-1}\eta(x_j) = \pi_{r,1} \sum_{j=0}^{r-2} \rho d'T^{r-j-1}\eta(x_j) \\ &= \pi_{r,1}\Phi \sum_{j=0}^{r-2} \delta_{r-j}(\tilde{x}_j) = \Phi_r \tilde{d}_r(\tilde{\zeta}), \end{aligned}$$

completing the proof of the theorem. □

Corollary 8.4. *Let Y be the bar resolution and (E_r, d_r) the LHS spectral sequence constructed using Y.*

(i) *There exist maps $\nu_n[\sigma_1|\cdots|\sigma_n] \in \hom_{kH}(P, P^{\sigma_1\cdots\sigma_n})_{n-1}$ ($n \geq 1$, $\sigma_1, \dots, \sigma_n \in Q$) which satisfy $\epsilon\nu_1[\sigma_1] = \epsilon$ and equation (1.2).*

(ii) *Let $(\tilde{E}_r, \tilde{d}_r)$ be the spectral sequence of the bigraded complex (A, δ), where $A = \operatorname{Hom}_k(\bar{Y}, \operatorname{Hom}_{kH}(P, M))$ and δ is defined by*

$$\delta_r(f)[\sigma_1|\cdots|\sigma_{p+r}] = (-1)^\gamma \sigma_1\cdots\sigma_r \circ f[\sigma_{r+1}|\cdots|\sigma_{p+r}] \circ \nu_r[\sigma_r^{-1}|\cdots|\sigma_1^{-1}],$$

where $\gamma = r(r+1)/2 + pr + q$, $f \in A^{p,q}$ and $r \geq 2$. Then the map $\Phi\colon \tilde{E}_1 \to E_1$ induces an isomorphism $\Phi_r\colon \tilde{E}_r \to E_r$ for all $r \geq 1$.

Proof. By setting

$$t[\sigma_1|\cdots|\sigma_n] = (-1)^{n(n+1)/2} \sigma_1\cdots\sigma_n \otimes \nu_n[\sigma_n^{-1}|\cdots|\sigma_1^{-1}] \in \hat{\mathfrak{A}}_{n-1},$$

it is easily seen that the existence of ν_n is equivalent to the existence of a twisting cochain associated to the extension, so (i) follows from Theorem 5.1, and (ii) is just a restatement of Theorem 8.1. □

Let us see how Corollary 8.4 implies the result, mentioned in Section 1, that if there is an action of Q on P which commutes with the differentials and the augmentation, and satisfies $\sigma(hx) = (\sigma h \sigma^{-1})\sigma(x)$ ($h \in H$, $x \in P$), then $E_2 = E_\infty$. Indeed, if this is the case, we may set $\nu_1[\sigma](x) = \sigma(x)$. We then have $\nu_1[\sigma] \circ \nu_1[\tau] = \nu_1[\sigma\tau]$, so we may take $\nu_2 = 0$. It follows by induction on n that we may take $\nu_n = 0$ for $n \geq 2$: suppose $\nu_i = 0$ for $2 \leq i \leq n-1$; then the right side of equation (1.2) vanishes, and we may therefore set $\nu_n = 0$. Hence $\delta_n = 0$ for $n \geq 2$, and therefore $\tilde{E}_2 = \tilde{E}_\infty$, so the same is true in the LHS spectral sequence.

We can also see that if H acts trivially on M, then all the differentials into the horizontal edge vanish. To do this we must first make the following observation: throughout we have assumed that $P \to k$ is the minimal resolution and $\operatorname{rad}_{kH}(M) = 0$. This hypothesis was needed to ensure that the differentials in $\operatorname{Hom}_{kH}(P, M)$ vanish, so that the latter is $H^*(H, M)$, and this was all that was required in the proofs. If H acts trivially on M, we can increase the size of P slightly so that in degree 0 we have kH. To do this, let W be a complement of the projective cover of k in kH. Then the projective cover of k is not a summand of W. Form the exact complex which in degrees 0 and 1 is W (and zero elsewhere) and which has the identity map for the differential. Replace P with the direct sum of P and this complex, and we have a projective resolution of the required form which is kH is degree 0. But now there is an obvious action of Q on P_0 induced by the action of Q on H, and we use this to define each $\nu_1[\sigma]$ on P_0. It follows that we can choose all the $\nu_2[\sigma|\tau]$ to vanish on P_0, and then an inductive proof like the one above shows we can take all $\nu_n[\sigma_1|\cdots|\sigma_n]$ $(n \geq 2)$ to vanish on P_0. Thus all the δ_n $(n \geq 2)$ which land in the horizontal edge vanish, so the same is true for the $\tilde{d}_n$, and therefore the d_n.

9. Examples

Now suppose that k is a field of characteristic 2, $Q = \langle b \mid b^2 = 1\rangle$, and $G = H \rtimes Q$. Let $Y \to k$ be the minimal kQ-resolution, say $Y_n = kQy_n$, $\partial(y_{n+1}) = (b+1)y_n$, and $\Delta(y_n) = \sum_{i=0}^{n} y_i \otimes b^i y_{n-i}$ $(n \geq 0)$. As we saw in Example 4.5, $(Y, \partial, \Delta, \imath)$ is a special resolution, where $\imath\colon \bar{Y} \to Y$ is defined by $\imath(\bar{y}_n) = y_n$. In this case, the twisting cochain formula becomes

$$d_{n-1}t_n = \sum_{i=1}^{n-1} t_i \cup t_{n-i}, \qquad n \geq 2, \tag{9.1}$$

as $\bar{\partial} = 0$. Thus the calculation of t reduces to finding maps $\alpha_n \in \hom_{kH}(P, P^{b^n})_{n-1}$ $(n \geq 1)$ satisfying

$$\begin{aligned} &\text{(i) } \epsilon\alpha_1 = \epsilon, \\ &\text{(ii) } \partial\alpha_2 + \alpha_2\partial = \alpha_1\alpha_1 + 1_P, \\ &\text{(iii) } \partial\alpha_n + \alpha_n\partial = \sum_{i=1}^{n-1} \alpha_i\alpha_{n-i}, \quad \text{for } n \geq 3. \end{aligned} \tag{9.2}$$

For given such maps, we may define $t_n(\bar{y}_n) = b^n \otimes \alpha_n \in \hat{\mathfrak{A}}_{n-1}$, and then t satisfies (9.1). We now have

$$A^{p,q} = \operatorname{Hom}_k(\bar{Y}_p, \operatorname{Hom}_{kH}(P_q, M)) \cong \operatorname{Hom}_{kH}(P_q, M),$$

and if $\zeta \in A^{p,q}$ corresponds to $f \in \operatorname{Hom}_{kH}(P_q, M)$, then $\delta_r(\zeta)$ corresponds to

$$b^r \circ f \circ \alpha_r \in \operatorname{Hom}_{kH}(P_{q-r+1}, M).$$

Now suppose that in addition we have $M = k$, $H = \langle a \mid a^{2^{m-1}}\rangle$ for some $m > 3$, and that $bab = a^d$ $(0 \leq d < 2^{m-1})$. There are three interesting cases to consider: (i) $d = 2^{m-1} - 1$, so G is the dihedral group of order 2^m, (ii) $d = 2^{m-2} - 1$, in which case G is the semidihedral group of order 2^m, and (iii) $d = 2^{m-2} + 1$, in which case $G = M_m(2)$ (in the notation of Gorenstein [7]). Together with the direct product, these represent the four isomorphism classes of groups which are split extensions of a cyclic group of order 2 by a cyclic group of order 2^{m-1} (cf. [7, Corollary 5.4.2]).

In any case, we have (say) $P_n = kHe_n$ $(n \geq 0)$, and $\partial(e_n) = (a+1)e_{n-1}$ if n is positive and even, and $\partial(e_n) = Ne_{n-1}$, where $N = \sum_{i=1}^{2^{m-1}} a^i$, if n is odd. Since $H^n(H, k) = k$ $(n \geq 0)$, Q acts trivially on $H^*(H, k)$, and $E_1 = E_2 = H^*(Q, k) \otimes H^*(H, k)$, which is depicted in the left hand side of Figure 1 (ignoring for the moment the arrows). The integers in the figure represent the k-dimension at each point, and the generators x_1, y, and x_2 are chosen in the natural way; for example, x_2 corresponds to the map $f \in \operatorname{Hom}_{kH}(P_2, k)$, where $f(e_2) = 1$.

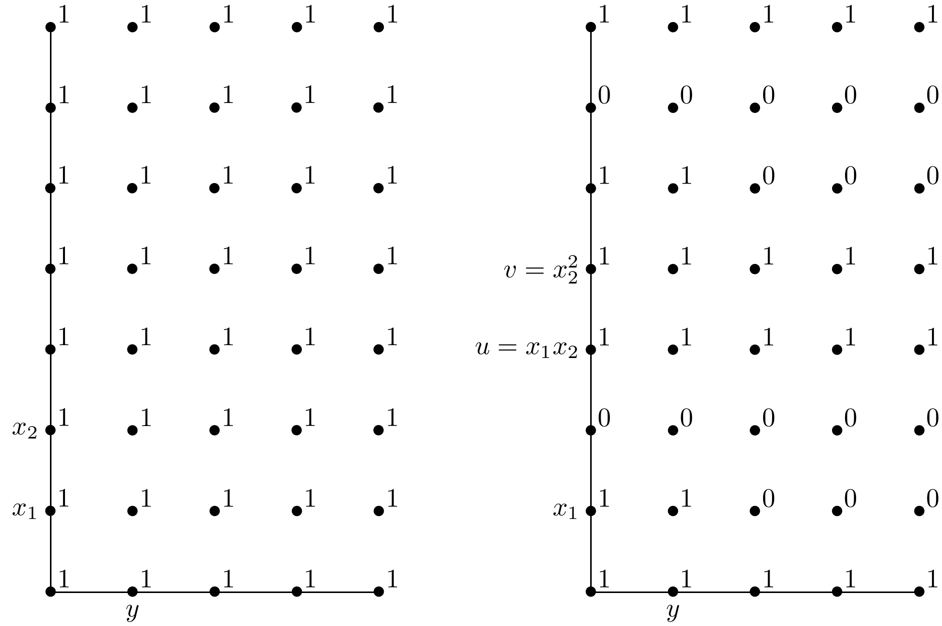

FIGURE 1. Spectral sequence for $\mathbb{Z}/2^{m-1} \rtimes \mathbb{Z}/2$.

Now if we let

$$\alpha_1(e_n) = \begin{cases} e_n & \text{if } n \text{ is even,} \\ \sum_{i=0}^{d-1} a^i e_n & \text{if } n \text{ is odd,} \end{cases}$$

an easy calculation shows that α_1 commutes with the differentials and that $\epsilon\alpha_1 = \epsilon$. To find a suitable α_2, we first calculate $1_P + \alpha_1^2$. Clearly this maps e_n to 0 for even n, while for n odd,

$$\alpha_1^2(e_n) = \alpha_1\left(\sum_{i=0}^{d-1} a^i e_n\right) = \left(\sum_{i=0}^{d-1} a^{di}\right)\left(\sum_{j=0}^{d-1} a^j\right) e_n = \kappa e_n,$$

where $\kappa = 1$ in the dihedral case, and $\kappa = N + 1$ in the other two cases.

Hence in the dihedral case, we have $\alpha_1^2 + 1_P = 0$, i.e. we have an actual action of Q on P, and as we have seen this implies $E_2 = E_\infty$.

In the other two cases, we may define α_2 by

$$e_{4i} \mapsto 0, \qquad e_{4i+1} \mapsto e_{4i+2}, \qquad e_{4i+2} \mapsto N_2 e_{4i+3}, \qquad e_{4i+3} \mapsto 0,$$

for $i \geq 0$, where

$$N_2 = N/(a+1) = \sum_{j=0}^{2^{m-2}-1} a^{2j},$$

and a quick calculation verifies that (9.2) is satisfied for $n = 2$. Hence by Theorem 8.1, $d_2(x)$ corresponds to $f \circ \alpha_2 \colon e_1 \mapsto e_2 \mapsto 1$, i.e. $d_2(x_2) = x_1 y^2$. Similarly, $d_2(x_1) = 0$, since $\alpha_2(e_0) = 0$. (This also follows from the fact that all differentials into the horizontal edge vanish in the case of a split extension.) This determines the effect of d_2 on all of E_2, as d_2 is a derivation. Finally, the positions of the generators of E_3 reveal that $E_3 = E_\infty$ (see Figure 1).

We remark that even though $d_r = 0$ for all $r \geq 3$ the same is not true of t_r. For $\alpha_1\alpha_2 + \alpha_2\alpha_1$ is defined by

$$e_{4i} \to 0, \qquad e_{4i+1} \mapsto \sum_{i=1}^{d-1} a^i e_{4i+2}, \qquad e_{4i+2} \mapsto \lambda e_{4i+3}, \qquad e_{4i+3} \mapsto 0,$$

for $i \geq 0$, where $\lambda = 0$ if $G = M_m(2)$, and $\lambda = N$ if G is semidihedral. Hence α_3, and therefore t_3, is nonzero.

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